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Integrable quantum spin chains, non-skew symmetric r-matrices and quasigraded Lie algebras

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Abstract

Using general non-skew symmetric classical r-matrices with a spectral parameter, we construct new quantum spin chains that generalize famous Gaudin spin chains. With the help of the special quasigraded Lie algebras, we construct new examples of the non-skew symmetric classical r-matrices with a spectral parameter and explicit examples of new commuting Gaudin-type hamiltonians.

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1. Introduction

In the last few years, interest has arisen in integrable Gaudin-like spin chains [1]. This is explained by the fact that several new connections between Gaudin magnets and other models of mathematical physics have been discovered. The first one is a connection between Gaudin spin chains and solutions of the Knizhnik–Zamolodchikov equations which associates Gaudin magnets with the conformal field theory [2,3]. The second is a connection between Gaudin spin chains and the BCS (Bardeen–Cooper–Schleiffer) model in one dimension [4,5]. These facts give new motivation to investigating the Gaudin-like spin chains. Known examples of Gaudin spin chains are connected with the skew symmetric solutions of the classical Yang–Baxter equations [1,6,7]. All such solutions are classified [8] and are exhausted by rational, trigonometric and elliptic solutions.

In our previous paper [9] we constructed some new integrable classical spin chains starting from the general non-skew symmetric solutions of the generalized classical Yang–Baxter equations (a dualized form of the modified Yang–Baxter equations [10–13]) with values in semi-simple (reductive) Lie algebras $\mathfrak{g}$. In particular, we have constructed the second order (in spin variables) hamiltonians of these systems and showed that they are direct analogs of the famous Gaudin hamiltonians. We have called our systems “generalized Gaudin systems”.

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In the present paper we consider a problem of the quantization of the generalized Gaudin systems. The quantization of the basic (spin) variables is achieved by the standard procedure of substitution of the Lie–Poisson bracket on $\mathfrak{g}^{\oplus N}$ by commutator or, in other words, in passing from $(S(\mathfrak{g}^{\oplus N}), \{\ ,\ \})$ to $(\mathfrak{A}(\mathfrak{g}^{\oplus N}), [\ ,\ ])$ or some of its representations in the Hilbert space $\mathcal{H}$. Nevertheless, due to the problem of ordering, such a simple recipe does not solve the problem of commutativity of the classically Poisson-commutative integrals which are non-linear in basic variables. This problem is non-trivial also in the case of integrable systems connected with the linear r-matrix bracket, especially in the case of the general non-skew symmetric r matrices with the spectral parameters. At the present moment, no general approach to a proof of the quantum integrability of such the classically integrable systems is known. That is why we deal with this problem in the case of the our systems directly. We give a direct proof that, after symmetrization in spin variables, generalized Gaudin hamiltonians stay commutative also in the quantum case. They have the following form:

$$\hat{H}^l = \sum_{k=1,k\neq l}^{N} \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r^{\alpha\beta}(u_k,u_l)\hat{S}^k_\alpha \hat{S}^l_\beta + \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r_0^{\alpha\beta}(u_l,u_l)(\hat{S}^l_\alpha \hat{S}^l_\beta + \hat{S}^l_\beta \hat{S}^l_\alpha), \quad l \in \overline{1,N}. \tag{1}$$

Here, N is a number of sites, $\hat{S}^l_\alpha$, $\hat{S}^k_\beta$ are components of the generalized spin operators in the site l and k, correspondingly, that obey the commutation relations of $\mathfrak{g}^{\oplus N}$, $r^{\alpha\beta}(u_k,u_l)$ are the matrix elements of the $\mathfrak{g}\otimes\mathfrak{g}$-valued classical r matrix $r(u_k,u_l)$, $r_0^{\alpha\beta}(u_l,u_l)$ are the matrix elements of its regular part (see definition (8)) and u_k, $k \in \overline{1,N}$, are some fixed complex numbers. Note that our hamiltonians coincide with the Gaudin hamiltonians only in the case $r^{\alpha\beta}(u_k,u_l) = -r^{\beta\alpha}(u_l,u_k)$. In the other cases they contain, as is clear from their form, besides the usual term describing the interactions of the spins living at different sites an additional term describing the "self-action" of the spin living at the same cite.

In order to make our construction more interesting and concrete, we construct new non-skew symmetric solutions of the generalized classical Yang–Baxter equation and use them to produce new Gaudin-type systems according to the formula (1). In order to construct such solutions, we use the connection of the classical Yang–Baxter equations with the theory of the infinite-dimensional Lie algebras [11]. We rely on the fact that each infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$-valued functions that admits a decomposition into a direct sum of two subalgebras $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-$ (Kostant–Adler scheme) also admits a classical r-matrix that coincides with the kernel of the operator R [11,15]:

$$R = \frac{1}{2}(P_+ - P_-), \quad \text{where } P_\pm \text{ are the projection operators on the subalgebras } \widetilde{\mathfrak{g}}_\pm.$$

Hence, in order to construct new classical r-matrices, it is necessary to construct new Lie algebras $\tilde{\mathfrak{g}}$ admitting decomposition into the direct sum of two subalgebras. In our previous papers [16–19] we have proposed for this role special quasigraded Lie algebras $\widetilde{\mathfrak{g}}_F$ with the decomposition $\widetilde{\mathfrak{g}}_F = \widetilde{\mathfrak{g}}_{F+} + \widetilde{\mathfrak{g}}_{F-}$ realized as a loop Lie algebra $\mathfrak{g}(u^{-1},u)$ with the Lie bracket, deformed by some cocycle F. Its Lie subalgebra $\widetilde{\mathfrak{g}}_{F-}$ was independently constructed also in [20] as a complementary subalgebra to the Lie algebra of Taylor series $\mathfrak{g}((u))$ in the Lie algebra of Laurent power series $\mathfrak{g}(u^{-1},u))$. In the present paper we combine our previous ideas with the results of [20] and realize $\widetilde{\mathfrak{g}}_F$ as a quasigraded Lie subalgebra in the Lie algebra of the Loran power series $\mathfrak{g}(u^{-1},u))$ defined with the help of some map $\Phi(u) : \mathfrak{g}(u^{-1},u)) \to \mathfrak{g}(u^{-1},u))$, where $\Phi(u) = 1 + \sum_{k=1}^{\infty} \Phi_k u^k$, $\Phi_k : \mathfrak{g} \to \mathfrak{g}$, $F = \delta\Phi_1$, and δ is a standard coboundary operator on the Lie algebra $\mathfrak{g}$. Using this realization, we obtain new non-skew symmetric solutions of the generalized classical Yang–Baxter equation $r_\Phi(u,v)$. These solutions generalize the so-called "anisotropic" non-skew symmetric r-matrices $r_A(u,v)$ constructed in our previous paper [9]. The corresponding generalized Gaudin hamiltonians have the following form:

$$\hat{H}^l = \sum_{k=1,k\neq l}^{N} \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} \frac{(\Phi^{-1}(u_l)\Phi(u_k))^{\alpha\beta}}{(u_k-u_l)}\hat{S}^k_\alpha \hat{S}^l_\beta + \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} (\Phi^{-1}(u_l)\partial_{u_l}\Phi(u_l))^{\alpha\beta}(\hat{S}^l_\alpha \hat{S}^l_\beta + \hat{S}^l_\beta \hat{S}^l_\alpha). \tag{2}$$

We consider in detail examples of such hamilonians that correspond to several particular cases of the map $\Phi(u)$ and to r-matrix $r_\Phi(u,v)$. We show, in particular, that in the case $N = 1$, $\mathfrak{g} = so(3)$ among the hamiltonians (2) corresponding to the different maps $\Phi(u)$ there is the quantized hamiltonian of the usual anisotropic Euler top, and in the case $N = 2$, $\mathfrak{g} = so(3)$ there are hamiltonians of the quantized Steklov system on $so(3) + so(3)$. Hence, in the case $N = 1$, hamiltonian (1) generalizes a quantized hamiltonian of the usual Euler top, and in the case $N = 2$ two hamiltonians (1) are generalizations of the quantized Steklov hamiltonians.

The structure of the present paper as follows. In the first section we introduce the main definitions and notations, formulate the main theorem and some of its consequences. In the second section we consider the connection of the classical r-matrices with the infinite-dimensional Lie algebras, introduce special quasigraded Lie algebras $\mathfrak{g}_F$ in different realizations and construct r-matrices $r_\Phi(u,v)$. In the third section we consider examples of the generalized Gaudin hamiltonians (1) associated with r-matrix $r_\Phi(u,v)$.

2. Integrable spin chains and classical r-matrices

Let $\mathfrak{g}$ be a simple (reductive) Lie algebra. Let X_α, $\alpha = \overline{1, \dim \mathfrak{g}}$ be some basis in $\mathfrak{g}$ with the commutation relations:

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{\dim \mathfrak{g}} C^\gamma_{\alpha\beta} X_\gamma. \tag{3}$$

Then $\hat{S}^i_\alpha$, $\alpha = \overline{1, \dim \mathfrak{g}}$, $i = \overline{1, N}$ are linear operators in some Hilbert space that constitute a Lie algebra isomorphic to $\mathfrak{g}^{\oplus N}$ with the commutation relations:

$$[\hat{S}^i_\alpha, \hat{S}^j_\beta] = \delta^{ij} \sum_{\gamma=1}^{\dim \mathfrak{g}} C^\gamma_{\alpha\beta} \hat{S}^j_\gamma. \tag{4}$$

We will consider operators $\hat{S}_\alpha$ to be an α component of the "generalized spin operator". Then operators $\hat{S}^i_\alpha$ could be interpreted as an α component of the generalized spin operator living at the i-cite of the generalized spin chain.

Example 1. In the case $\mathfrak{g} = so(3)$, operators $\hat{S}_a$, $\alpha = \overline{1, 3}$, are components of the usual spin and operators $\hat{S}^i_\alpha$ have the usual $so(3)^{\oplus N}$ commutation relations:

$$[\hat{S}^i_\alpha, \hat{S}^j_\beta] = \delta^{ij} \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \hat{S}^j_\gamma. \tag{5}$$

We will be interested in a construction of the family of the second order (in "spin" variable $\hat{S}^i_\alpha$) mutually commuting operators $\hat{H}^j$, $j = \overline{1, N}$, that generalize standard Gaudin hamiltonians.

We will need the following definition:

Definition 1. A function of two complex variables $r(u_1, u_2)$ with values in the tensor square of the algebra $\mathfrak{g}$ is called a classical r-matrix if it satisfies the following "generalized" classical Yang–Baxter equation[1]:

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] - [r_{32}(u_3, u_2), r_{13}(u_1, u_3)], \tag{6}$$

where $r_{12}(u_1, u_2) \equiv \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha\beta}(u_1, u_2) X_\alpha \otimes X_\beta \otimes 1$ etc.

Remark 1. If the matrix $r(u_1, u_2)$ is "skew", i.e. $r_{12}(u_1, u_2) = -r_{21}(u_2, u_1)$, Eq. (6) passes into the usual classical Yang–Baxter equation:

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2) + r_{13}(u_1, u_3)]. \tag{7}$$

Due to the fact that all solutions of the Eqs. (7) are skew [21], each solution of (7) is also a solution of (6).

Let us fix N distinct points $\{u_k\}$ on the complex plane such that, in the neighborhood of these points, $r(u, v)$ possesses the following decomposition:

$$r(u, v) = \frac{\widehat{c}}{(u - v)} + r_0(u, v) \tag{8}$$

[1] This equation first appeared in somewhat different form in the papers [10] and [12]. In the present form it has appeared later in the paper [14].

where $r_0(u, v)$ is a regular in the neighborhood of the points $u = u_k$, $v = u_m$, where $k, m \in 1, N$, $\mathfrak{g} \otimes \mathfrak{g}$-valued function, $\widehat{c} \in \mathfrak{g} \otimes \mathfrak{g}$ is the tensor Casimir, i.e. $\widehat{c} = \sum_{\alpha,\beta} g^{\alpha\beta} X_\alpha \otimes X_\beta$, $g^{\alpha\beta}$ is the nondegenerate invariant metric on $\mathfrak{g}$.

The following theorem holds true:

Theorem 2.1. *Let* $r(u_i, u_j) \equiv \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha\beta}(u_i, u_j) X_\alpha \otimes X_\beta$ *be the classical r-matrix and* $r_0(u_i, u_j)$ *its regular part. Then the second order operators* $\hat{H}^l$, $l = 1, N$ *of the type:*

$$\hat{H}^l = \sum_{k=1, k\neq l}^{N} \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha\beta}(u_k, u_l) \hat{S}_\alpha^k \hat{S}_\beta^l + \frac{1}{2} \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r_0^{\alpha\beta}(u_l, u_l)(\hat{S}_\alpha^l \hat{S}_\beta^l + \hat{S}_\beta^l \hat{S}_\alpha^l) \tag{9}$$

constitute commutative subalgebra in the universal enveloping algebra $\mathfrak{A}(\mathfrak{g}^{\oplus N})$ *and all its representations.*

Idea of the Proof. We prove the theorem directly, showing that $[\hat{H}^k, \hat{H}^l] = 0\ \forall k, l, k \neq l$. In the proof, we use the generalized classical Yang–Baxter equation (6) and some of its consequences. The proof is straightforward but a bit lengthy, and is detailed in the Appendix.

Remark 2. In the case of the skew symmetric r-matrices, we have that $r_0^{\alpha,\beta}(u_l, u_l) = -r_0^{\beta,\alpha}(u_l, u_l)$ and, hence, the additional term in the corresponding hamiltonian $\hat{H}^l$ turns out to be zero and $\hat{H}^l$ coincides with the standard Gaudin hamiltonian.

Remark 3. In the classical limit, hamiltonian H^l acquires the form:

$$H^l = \sum_{k=1, k\neq l}^{N} \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha\beta}(u_k, u_l) S_\alpha^k S_\beta^l + \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r_0^{\alpha\beta}(u_l, u_l) S_\alpha^l S_\beta^l. \tag{10}$$

In this case, they Poisson-commute: $\{H^k, H^l\} = 0$ with respect to the natural Lie–Poisson bracket on the direct sum of N copies of $\mathfrak{g}^*$:

$$\{S_\alpha^i, S_\beta^j\} = \delta^{ij} \sum_{\gamma=1}^{\dim \mathfrak{g}} C_{\alpha\beta}^\gamma S_\gamma^j. \tag{11}$$

Their commutativity also follows from the representation:

$$H^k = \mathrm{res}_{u=u_k} c^2(L(u)), \tag{12}$$

where c^2 is a second order Casimir of $\mathfrak{g}$ defined with the help of the invariant tensor $g^{\alpha\beta}$: $c^2(L) = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} g^{\alpha\beta} L_\alpha L_\beta$ and the L-operator $L(u)$ is defined as follows [9]:

$$L(u) = \sum_{k=1}^{N} \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} S_\alpha^k r^{\alpha\beta}(u_k, u) X_\beta.$$

For the case of a general $\mathfrak{g}$, these hamiltonians do not form a complete family of commuting integrals and they should be completed by the following "higher" integrals:

$$I_n^{r,i} = \mathrm{res}_{u=u_i}(u - u_i)^n c^r(L(u)), \tag{13}$$

where $c^r(L)$ is a Casimir function on $\mathfrak{g}$ of order r.

A proof of the quantum commutativity of the "higher" integrals is a separate complicated problem even in the case of the standard skew symmetric r-matrices. This problem was approached for the case of the general skew symmetric r-matrices in [23] and solved for the case of the simplest rational r-matrix in [24].

Example 2. In the one-spin case ($N = 1$), our construction gives one the hamiltonian:

$$\hat{H}^1 = 1/2 \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r_0^{\alpha\beta}(u_1, u_1)(\hat{S}_\alpha^1 \hat{S}_\beta^1 + \hat{S}_\beta^1 \hat{S}_\alpha^1).$$

As will be clear from Example 16, these are generalizations of the quantized hamiltonian of the anisotropic Euler top on $so(3)$.

Example 3. In the case of two spins $\hat{S}^1$, $\hat{S}^2$ ($N = 2$), our construction yields the following two independent mutually commuting quantum hamiltonians:

$$\hat{H}^1 = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha\beta}(u_2, u_1)\hat{S}^2_\alpha \hat{S}^1_\beta + 1/2 \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r_0^{\alpha\beta}(u_1, u_1)(\hat{S}^1_\alpha \hat{S}^1_\beta + \hat{S}^1_\beta \hat{S}^1_\alpha),$$

$$\hat{H}^2 = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha\beta}(u_1, u_2)\hat{S}^1_\alpha \hat{S}^2_\beta + 1/2 \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r_0^{\alpha\beta}(u_2, u_2)(\hat{S}^2_\alpha \hat{S}^2_\beta + \hat{S}^2_\beta \hat{S}^2_\alpha).$$

As will be clear from Example 16, these hamiltonians are generalizations of the quantized commuting Steklov top hamiltonians on $so(3) \oplus so(3)$.

3. Classical *r* matrices and quasigraded Lie algebras

In this section we construct new non-skew classical r matrices with the spectral parameters starting from the special quasigraded Lie algebras, which are constructed in the second and third subsections of this section.

3.1. Classical r-matrices from special infinite-dimensional algebras

In this subsection we will recall briefly the Kostant–Adler scheme and its connection with the solutions of Eq. (6). Let $\widetilde{\mathfrak{g}}$ hereafter be some infinite-dimensional Lie algebra of $\mathfrak{g}$-valued functions of one complex variable u with values in a semisimple (reductive) Lie algebra $\mathfrak{g}$ and natural Lie bracket [,]. Now let the Lie algebra $\widetilde{\mathfrak{g}}$ possess the Kostant–Adler scheme, i.e. admit as a linear space decomposition into the direct sum of two subalgebras:

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-. \tag{14}$$

It is well known [22] that it is possible to equip the linear space $\widetilde{\mathfrak{g}}$ with new "direct difference" Lie brackets:

$$[X(u), Y(u)]_0 = [X_+(u), Y_+(u)] - [X_-(u), Y_-(u)], \tag{15}$$

where $X(u) = X_+(u) + X_-(u)$, $Y(u) = Y_+(u) + Y_-(u)$ and $X_\pm(u), Y_\pm(u) \in \widetilde{\mathfrak{g}}_\pm$. Let $P_\pm$ be two projection operators on the Lie subalgebras $\widetilde{\mathfrak{g}}_\pm$ correspondingly: $P_\pm(X(u)) = X_\pm(u)$. It is also well known [15] that this bracket could be represented in the form:

$$[X(u), Y(u)]_0 = [R(X(u)), Y(u)] + [X(u), R(Y(u))],$$

where operator

$$R = 1/2(P_+ - P_-) \tag{16}$$

satisfies the so-called modified Yang–Baxter equation [11].

Let us now assume that this R-operator possesses the kernel:

$$R(X)(u) = \oint_{u=0} (r_{12}(u, v), X_2(v))_2, \tag{17}$$

where $r_{12}(u, v)$ is $\mathfrak{g} \otimes \mathfrak{g}$-valued function of two complex variables, $X_2(v) \equiv 1 \otimes X_2$, and (,) is invariant non-degenerated bilinear form on $\mathfrak{g}$. It is possible to show (see [22]) that the function $r_{12}(u, v)$ satisfies the generalized Yang–Baxter equation (6).

Let $\widetilde{X^m_\alpha} \equiv \widetilde{X^m_\alpha}(u)$, where $m \in Z$, $\alpha \in 1, \dim \mathfrak{g}$ is a basis in $\widetilde{\mathfrak{g}}$ compatible with the decomposition (14), i.e. $\widetilde{X^m_\alpha} \in \widetilde{\mathfrak{g}}_+$ for $m \geq 0$ and $\widetilde{X^m_\alpha} \in \widetilde{\mathfrak{g}}_-$ for $m < 0$. Let $\widetilde{Y^m_\alpha} \equiv Y^m_\alpha(u)$ be a basis in the linear space $\widetilde{\mathfrak{g}}^*$ dual to the basis $\widetilde{X^m_\alpha}$ in $\widetilde{\mathfrak{g}}^*$ with respect to the natural pairing $\langle X(u), Y(u)\rangle = \mathrm{res}_{u=0}(X(u), Y(u))$. In this case, the kernel r of the operator R can be represented as follows:

$$r_{1,2}(u,v) = 1/2\left(\sum_{\alpha=1}^{\dim\mathfrak{g}}\sum_{m\geq 0}\widetilde{X_\alpha^m}(u)\otimes\widetilde{Y_\alpha^m}(v) - \sum_{\alpha=1}^{\dim\mathfrak{g}}\sum_{m<0}\widetilde{X_\alpha^m}(u)\otimes\widetilde{Y_\alpha^m}(v)\right). \tag{18}$$

This formula will be the basis in our search for new solutions of Eq. (6).

3.2. Quasigraded Lie algebras with Kostant–Adler decomposition

As is well known [22], many classical r-matrices are connected with the graded algebras and their decompositions into a direct sum of two subalgebras. An important observation made in our previous papers [16–19] was that some quasigraded Lie algebras also admit such decompositions and possess the corresponding r-matrix. In what follows, we generalize this construction. Let us begin with the following definition [25]:

Definition. An infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ is called $\mathbb{Z}$-quasigraded of type (p,q) if it admits a decomposition:

$$\widetilde{\mathfrak{g}} = \sum_{j\in\mathbb{Z}}\mathfrak{g}_j, \quad \text{such that } [\mathfrak{g}_i,\mathfrak{g}_j]\subset\sum_{k=-p}^{q}\mathfrak{g}_{i+j+k}.$$

The following important observation [16–19] holds true:

Proposition 3.1. *Let $\widetilde{\mathfrak{g}}$ be $\mathbb{Z}$-quasigraded of type $(0,1)$, or $(1,0)$. Then $\widetilde{\mathfrak{g}}$ admits decomposition (14) with $\widetilde{\mathfrak{g}}_\pm = \sum_{j\in\mathbb{Z}_\pm}\mathfrak{g}^j$.*

Let $\mathfrak{g}$ be some finite-dimensional Lie algebra with the Lie bracket $[\ ,\ ]$. One of the possible realizations of the above quasigraded Lie algebra is a loop space $\mathfrak{g}(u^{-1},u) = \mathfrak{g}\times Pol(u,u^{-1})$ (we will also use notation $\mathfrak{g}(u^{-1},u)$) for its extension with Laurent power series) with the specially defined bracket. Indeed, let $X_\alpha^i = X_\alpha\otimes u^i \in \mathfrak{g}^i$ be a basis in $\widetilde{\mathfrak{g}}$ where X_α is a basic element of semisimple Lie algebra $\mathfrak{g}$. Then it is easy to see that, on such basic elements, the required bracket should be written as follows [19]:

$$[X_\alpha^i, X_\beta^j]_F = [X_\alpha, X_\beta]^{i+j} + F(X_\alpha, X_\beta)^{i+j+1}, \tag{19}$$

where $[X_\alpha, X_\beta]^{i+j}\equiv[X_\alpha, X_\beta]\otimes u^{i+j}$, $F(X_\alpha,X_\beta)^{i+j+1}\equiv F(X_\alpha,X_\beta)\otimes u^{i+j+1}$ and $F(\ ,\ )$ a is a skew symmetric map $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying two requirements [19]:

$$\sum_{\text{c.p. }\{\alpha,\beta,\gamma\}}[F(X_\alpha,X_\beta),X_\gamma] + F([X_\alpha,X_\beta],X_\gamma) = 0, \tag{20a}$$

$$\sum_{\text{c.p. }\{\alpha,\beta,\gamma\}}F(F(X_\alpha,X_\beta),X_\gamma) = 0. \tag{20b}$$

Condition (20b) is exactly the Jacobi identity for the Lie brackets defined by cochains F: $[\ ,\ ]_1 = F(\ ,\ )$. Hence we have reduced a construction of "Kostant–Adler-admissible" quasigraded Lie algebras to a task of finding a pair of two Lie brackets on the vector space $\mathfrak{g}$ satisfying Eq. (20a). Eq. (20a) is easily seen to be equivalent to the compatibility condition for the Lie brackets $[\ ,\ ]$ and $[\ ,\ ]_1$. The problem of constructing the compatible Lie brackets on $\mathfrak{g}$ was investigated in [26,27,20]. We will use the following observation [20]:

Proposition 3.2. *Let a Lie algebra $\mathfrak{g}$ be such that $H^2(\mathfrak{g},\mathfrak{g}) = 0$, in particular let $\mathfrak{g}$ be semisimple. Denote by δ a standard co-boundary operator on the $\mathfrak{g}$-valued polylinnear forms. Then*

(i) *There is a linear map $\Phi_1:\mathfrak{g}\to\mathfrak{g}$ such that*

$$F(X,Y) = \delta\Phi_1(X,Y)\equiv[\Phi_1(X),Y] + [X,\Phi_1(Y)] - \Phi_1([X,Y])$$

solves condition (20a).

(ii) *Condition (20b) is equivalent to the following equation on the map Φ_1:*

$$\Phi_1(\delta\Phi_1(X,Y)) - [\Phi_1(X), \Phi_1(Y)] = \delta\Phi_2(X,Y), \tag{21}$$

for some linear map $\Phi_2 : \mathfrak{g} \to \mathfrak{g}$.

Example 4. In the case $\Phi_2 = 0$, this equation coincide with the condition of triviality of Nijenhuis curvature for the tensor Φ_1:

$$\Phi_1(\delta\Phi_1(X,Y)) - [\Phi_1(X), \Phi_1(Y)] = 0. \tag{22}$$

The simplest solution of the Eqs. (21) and (22) corresponds to the case $\Phi_1^2 = 0$. This reduces Eq. (21) to the classical Yang–Baxter equation:

$$\Phi_1([\Phi_1(X), Y] + [X, \Phi_1(Y)]) - [\Phi_1(X), \Phi_1(Y)] = 0, \tag{23}$$

i.e. any nilpotent solutions of the constant Yang–Baxter equations provides us with solutions of Eqs. (21) and (22).

Example 5. Let us consider the case of one of the classical matrix Lie algebras $\mathfrak{g} = gl(n), so(n), sp(n)$ and two cochains Φ_1, Φ_2 given by the formula:

$$\Phi_1(X) = \frac{1}{2}(AX + XA), \quad \Phi_2(X) = \frac{1}{4}\left(AXA - \frac{1}{2}(A^2X + XA^2)\right)$$

where A is a matrix such that $1/2(AX + XA) \in \mathfrak{g}$ and arbitrary otherwise. By direct calculations, one can show that such cochains satisfy Eq. (21). Direct calculations also show that $\delta\Phi_1(X,Y) = XAY - YAX$, and we arrive at the cocycle F_A from [26] which is used extensively in our previous papers [16–19].

Remark 4. Let us explain the relation of the compatible Lie brackets that could be constructed using Proposition 3.2 to the classification of the compatible Lie brackets given in [26]. In [26] so-called “closed” pencils of the compatible Lie brackets were classified on the finite-dimensional linear spaces. The condition of “closedness” is very restrictive. It is not true for all pencils of the compatible Lie brackets. Indeed, in the case when the linear space coincides with the classical matrix Lie algebras with the usual matrix commutator, the corresponding second Lie bracket that is compatible with it, according to the classification of [26], is only the bracket F_A from Example 5. But the corresponding cochains Φ_1 and Φ_2 of Example 5 do not exhaust the set of all solutions of Eq. (21). For example, the cochains Φ_1 from Example 4 (in the case rank $\mathfrak{g} \geq 2$) do not fall into the category of the cochains from Example 5 and the corresponding Lie pencil is not “closed” in the sence of [26].

Definition. We denote infinite-dimensional Lie algebras constructed via the cocycle $F = \delta\Phi_1$ by $\widetilde{\mathfrak{g}_F}$. The Lie algebras $\widetilde{\mathfrak{g}_F}$ are quasigraded of the type (0, 1) by the very construction and, by virtue of Proposition 3.1, admits decomposition into the direct sum of two subalgebras: $\widetilde{\mathfrak{g}_F} = \widetilde{\mathfrak{g}_{F+}} + \widetilde{\mathfrak{g}_{F-}}$.

Remark 5. We have constructed a special quasigraded Lie algebra $\widetilde{\mathfrak{g}_F}$ as a loop Lie algebra $\mathfrak{g}(u^{-1}, u)$ with the modified Lie bracket. In the next section we will realize Lie algebra $\widetilde{\mathfrak{g}_F}$ as a special quasigraded subalgebra of the Lie algebra $\mathfrak{g}(u^{-1}, u))$ of formal Laurent power series, but with a standard Lie bracket in the tensor product. This will be necessary in order to construct the corresponding classical r-matrix using formula (18).

3.3. Functional realization of $\widetilde{\mathfrak{g}_F}$ and $\widetilde{\mathfrak{g}_F}^*$

Let us now consider a Lie algebra of Laurent power series $\mathfrak{g}(u^{-1}, u))$. Let us also consider the linear spaces $\widetilde{\mathfrak{g}_{\Phi}}_-$, $\widetilde{\mathfrak{g}_{\Phi}}_+$, $\widetilde{\mathfrak{g}_{\Phi}} \in \mathfrak{g}(u^{-1}, u))$ spanned by finite linear combination of the following formal basic elements:

$$\widetilde{X}_a^n = (1 + u\Phi_1 + u^2\Phi_2 + u^3\Phi_3 + \cdots) \cdot (X_\alpha u^n) \equiv \Phi(u) \cdot (X_\alpha u^n), \tag{24}$$

where $n < 0$, $n \geq 0$ and $n \in \mathrm{Z}$ correspondingly.

As was noticed in the paper [20], the following lemma holds true:

Lemma 3.1. *Let $H^2(\mathfrak{g},\mathfrak{g}) = 0$ and the cochains Φ_1, Φ_2 satisfy Eq. (21). Then the coefficients Φ_k, $k > 2$ could be chosen in such a way that $\Phi(u)([X,Y] + u\delta\Phi_1(X,Y)) = [\Phi(u)(X), \Phi(u)(Y)]$. Coefficients of $\Phi(u)$ are defined up to the automorphism $g(u)$ of $\mathfrak{g}(u^{-1},u))$ of the form $g(u) = 1 + \sum_{k=1}^{\infty} g_k u^k$, $g_k \in \mathrm{Aut}(\mathfrak{g})$.*

Remark 6. Cochains Φ_k, $k > 2$ are (modulo the ambiguity connected with the automorphism $g(u)$) calculated via the following recurrence formula:

$$\delta\Phi_k(X,Y) = -\Phi_{k-1}(\delta\Phi_1(X,Y)) + \sum_{l=1}^{k-1}[\Phi_l(X), \Phi_{k-l}(Y)]. \tag{25}$$

From this, it follows that if $\Phi_2 = 0$, then $\Phi_k = 0$, $\forall k > 2$.

Using the above lemma it is straightforward to prove the following proposition:

Proposition 3.3. *Let $H^2(\mathfrak{g},\mathfrak{g}) = 0$ and the cochains Φ_1, Φ_2 satisfy Eq. (21). Then the linear spaces $\widetilde{\mathfrak{g}_{\Phi}}$, $\widetilde{\mathfrak{g}_{\Phi+}}$, $\widetilde{\mathfrak{g}_{\Phi-}}$ form closed Lie subalgebras in the Lie algebra $\mathfrak{g}(u^{-1},u))$. Moreover, these algebras are quasigraded and the following isomorphisms hold: $\widetilde{\mathfrak{g}_{\Phi}} \simeq \widetilde{\mathfrak{g}_F}$, $\widetilde{\mathfrak{g}_{\Phi+}} \simeq \widetilde{\mathfrak{g}_{F+}}$, $\widetilde{\mathfrak{g}_{\Phi-}} \simeq \widetilde{\mathfrak{g}_{F-}}$.*

Remark 7. The fact that $\widetilde{\mathfrak{g}_{\Phi-}}$ forms closed Lie algebra was noticed also in [20]. But the possibility to extend it by the algebra $\widetilde{\mathfrak{g}_{\Phi+}}$ to the full algebra $\widetilde{\mathfrak{g}_{\Phi}}$ and quasigraded character of all these algebras was not noticed and acknowledged there.

Remark 8. Note that, for the case when $\Phi_k = 0, \forall k > 1$, algebra $\widetilde{\mathfrak{g}_{\Phi}}$ is realized as a quasigraded subalgebra of the loop algebra $\mathfrak{g}(u,u^{-1})$ spanned over the monomials

$$\widetilde{X}_a^n = (X_\alpha u^n + \Phi_1(X_\alpha)u^{n+1}).$$

Note that, even in this case, $\widetilde{\mathfrak{g}_{\Phi}}$ is not isomorphic to the loop algebra $\mathfrak{g}(u,u^{-1})$, because the image of the inverse homomorphism $\Phi(u)^{-1}$ is a formal power series and does not, rigorously speaking, belong to the space of Laurent polynomials. Nevertheless, the isomorphism $\widetilde{\mathfrak{g}_{\Phi}} \simeq \mathfrak{g}(u^{-1},u)$ holds for the nilpotent maps Φ_1 ($\Phi_k = 0, k > 1$), in particular when $\Phi_1^2 = 0$.

Example 6. Let us consider the case of the classical matrix Lie algebras $\mathfrak{g} = gl(n), so(n), sp(n)$ and two cochains Φ_1, Φ_2 defined as in Example 5. By a direct calculation, using the formula (25) it is possible to show that the map $\Phi(u)$ in this case is given by the formula:

$$\Phi(u)(X) = \sqrt{1+Au}\,X\sqrt{1+Au},$$

i.e. we obtain an "irrational" realization of the Lie algebra $\widetilde{\mathfrak{g}_F} \simeq \widetilde{\mathfrak{g}}_A$ used in papers [16–19].

Having in mind the explicit functional realization $\widetilde{\mathfrak{g}_{\Phi}}$ of the algebra $\widetilde{\mathfrak{g}_F}$, we can easily write an explicit form of the dual space and coadjoint representation. Let us define a pairing between $X(u) \in \widetilde{\mathfrak{g}_{\Phi}}$ and $Y(u) \in \widetilde{\mathfrak{g}_{\Phi}}^*$ in the standard way:

$$\langle X(u), Y(u)\rangle = \oint_{u=0} (X(u), Y(u)),$$

where (,) is a nondegenerate invariant form on $\mathfrak{g}$.

The basic elements of the dual space $\widetilde{\mathfrak{g}_{\Phi}}^*$ under such a choice of the pairing have the form:

$$\widetilde{Y}_a^{-n} = (1 + u\Phi_1^* + u^2\Phi_2^* + \cdots)^{-1} \cdot (X^\alpha u^{n-1}), \tag{26}$$

where X^α is a basic element of the dual space: $(X_\beta, X^\alpha) = \delta_\beta^\alpha$ and Φ_k^* is a linear operator on $\mathfrak{g}$ dual to Φ_k with respect to the scalar product (,): $(\Phi_k(X), Y) = (X, \Phi_k^*(Y))$.

Example 7. In the case $\Phi_2 = 0$ and $\Phi_1^2 = 0$, this formula acquires a more simple form:

$$\widetilde{Y}_a^{-n} = (1 + u\Phi_1^*)^{-1} \cdot (X^\alpha u^{n-1}) = (1 - u\Phi_1^*) \cdot (X^\alpha u^{n-1}). \tag{27}$$

Example 8. In the case of the classical matrix Lie algebras $\mathfrak{g} = gl(n), so(n), sp(n)$ and two cochains Φ_1, Φ_2 defined as in Example 5, we have that:

$$\widetilde{Y}_a^{-n} = \sqrt{(1 + Au)^{-1}} X^\alpha u^{n-1} \sqrt{(1 + Au)^{-1}}. \tag{28}$$

Remark 9. It is evident from the explicit form of the basic elements of the algebra and dual space that $\widetilde{\mathfrak{g}_\Phi}^*$ does not coincide with $\widetilde{\mathfrak{g}_\Phi}$ as a linear space. It is possible to show that they also do not coincide as $\widetilde{\mathfrak{g}_\Phi}$-modules. This means that the corresponding r-matrix is not skew symmetric.

3.4. New non-skew classical r-matrices

Now, using the results of the previous sections, it is easy to calculate new solutions of the classical YB-equation that correspond to the algebra $\widetilde{\mathfrak{g}_F}$.

Theorem 3.1. *Let $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ and the map $\Phi(u) : \mathfrak{g}(u^{-1}, u)) \to \mathfrak{g}(u^{-1}, u))$: $\Phi(u) = \sum_{k=0}^{\infty} \Phi_k u^k$ such that $\Phi_0 = 1$, Φ_1 and Φ_2 satisfy Eq. (21) and coefficients Φ_k are chosen according to Lemma 3.1, then:*

(i) *the function $r_\Phi(u, v)$ of two complex variables:*

$$r_\Phi(u, v) = \frac{\sum_{\alpha=1}^{\dim \mathfrak{g}} \Phi(u) \cdot X_\alpha \otimes (\Phi(v)^{-1})^* \cdot X^\alpha}{(u - v)} \tag{29}$$

satisfies the generalized classical Yang–Baxter equation (6).

(ii) *The regular part of the classical r-matrix $r_\Phi(v, v)$ has the following form:*

$$(r_\Phi)_0(v, v) = \sum_{\alpha=1}^{\dim \mathfrak{g}} \partial_v \Phi(v) \cdot X_\alpha \otimes (\Phi(v)^{-1})^* \cdot X^\alpha. \tag{30}$$

Proof. Proof of item (i) follows from the application of formula (18) and the explicit form of the basic elements (24) and (26). Item (ii) is proved by direct verification. □

Example 9. For the case of the nilpotent solutions of Eq. (21) such that $\Phi_1^2 = 0$, $\Phi_2 = 0$, we see that the corresponding r-matrix is "rational":

$$r_{\Phi_1}(u, v) = \frac{\sum_{\alpha=1}^{\dim \mathfrak{g}} (1 + u\Phi_1) \cdot X_\alpha \otimes (1 - v\Phi_1^*) \cdot X_\alpha^*}{(u - v)}. \tag{31}$$

Taking into account that $\Phi_1^2 = 0$ and making simple transformations, we obtain:

$$r_{\Phi_1}(u, v) = \frac{\hat{c}}{(u - v)} + \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} \Phi_1^{\alpha\beta} X_\alpha \otimes X_\beta, \tag{32}$$

where $\Phi_1^{\alpha\beta} = g^{\beta\gamma} (\Phi_1)_\gamma^\alpha$ and $g^{\beta\gamma} = (X^\beta, X^\gamma)$. Evidently, the regular part of the r-matrix has the form $r_0 = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} \Phi_1^{\alpha\beta} X_\alpha \otimes X_\beta$. It is easy to show that it satisfies the constant Yang–Baxter equation: $[(r_0)_{12}, (r_0)_{13}] = [(r_0)_{23}, (r_0)_{12}] - [(r_0)_{32}, (r_0)_{13}]$.

Example 10. Let us consider classical matrix Lie algebras $\mathfrak{g} = gl(n)$, $\mathfrak{g} = so(n)$, $\mathfrak{g} = sp(n)$ and cochains Φ_1 and Φ_2 defined in Example 5. In this case, using Examples 7 and 8 we obtain that formula (29) acquires the following form:

$$r_\Phi(u, v) = \frac{1}{(u - v)} \sum_{\alpha=1}^{\dim \mathfrak{g}} \sqrt{1 + Au} X_\alpha \sqrt{1 + Au} \otimes (\sqrt{1 + Av})^{-1} X_\alpha^* (\sqrt{1 + Av})^{-1}. \tag{33}$$

This is exactly an "irrational" classical r-matrix $r_{\Phi}(u,v) \equiv r_A(u,v)$ described in our previous paper [9]. Its "regular" part is given by the following formula:

$$r_{\Phi}(v,v) = \left(\sum_{\alpha=1}^{\dim \mathfrak{g}} \partial_v(\sqrt{1+Av}X_\alpha\sqrt{1+Av}) \otimes (\sqrt{1+Av})^{-1}X_\alpha^*(\sqrt{1+Av})^{-1}\right).$$

Example 11. Let us consider in detail the r-matrices from the previous example, but for the special cases of the diagonal matrices A: $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$. Letting $X_\alpha = X_{ij}$ be the standard "matrix" basis in these algebras, then formula (29) acquires the following simple form:

$$r_A(u,v) = \frac{1}{(u-v)}\sum_{ij=1}^{n}\sqrt{\frac{(1+a_iu)(1+a_ju)}{(1+a_iv)(1+a_jv)}}X_{ij}\otimes X_{ji}. \tag{34}$$

Its "regular" part also has a very nice form:

$$(r_A)_0(v,v) = \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{a_i}{(1+a_iv)} + \frac{a_j}{(1+a_jv)}\right)X_{ij}\otimes X_{ji}. \tag{35}$$

Example 12. Let $\mathfrak{g} = so(3)$. Then, introducing the new basis $X_i = \epsilon_{ijk}X_{jk}$ and using the previous example, we obtain the following explicit form of the irrational $so(3)$ r-matrix:

$$r_A(u,v) = \prod_{i=1}^{3}\sqrt{\frac{(1+a_iu)}{(1+a_iv)}}\sum_{k=1}^{3}\sqrt{\frac{(1+a_kv)}{(1+a_ku)}}\frac{X_k\otimes X_k}{(u-v)}. \tag{36}$$

Its "regular" part is written as:

$$(r_A)_0(v,v) = \frac{1}{2}\prod_{i=1}^{3}(1+a_iv)^{-1}\sum_{k=1}^{3}\left(\frac{a_1a_2a_3}{a_k}v - a_k\right)(1+a_kv)X_k\otimes X_k. \tag{37}$$

4. Examples of new integrable quantum spin chains

In this section, using the classical r-matrices constructed in the previous section will explicitly obtain Hamiltonians of the new integrable quantum spin chains (9).

From Theorems 2.1 and 3.1, the next corollary follows:

Corollary 4.1. *Let a Lie algebra $\mathfrak{g}$ be semisimple (reductive) and the map $\Phi(u) : \mathfrak{g} \to \mathfrak{g}$ satisfy the conditions of Theorem 3.1. Let map $\Phi(u)$ be correctly defined in the neighborhood of the points $u = u_l, l \in \overline{1,N}$ and possess formal inverse:* $\det \Phi(u_l) \neq 0$. *Then the operators $\hat{H}^l$:*

$$\hat{H}^l = \sum_{k=1,k\neq l}^{N}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}\frac{(\Phi^{-1}(u_l)\Phi(u_k))^{\alpha\beta}}{(u_k-u_l)}\hat{S}_\alpha^k\hat{S}_\beta^l + \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}(\Phi^{-1}(u_l)\partial_{u_l}\Phi(u_l))^{\alpha\beta}(\hat{S}_\alpha^l\hat{S}_\beta^l + \hat{S}_\beta^l\hat{S}_\alpha^l), \quad l = \overline{1,N} \tag{38}$$

constitute a commutative family in the universal enveloping algebra $\mathfrak{A}(\mathfrak{g}^{\oplus N})$ and all its linear representations.

Let us consider concrete examples of this formula for the concrete r-matrices r_ϕ:

Example 13. Let $\mathfrak{g}$ be an arbitrary semisimple Lie algebra and r-matrix $r_\phi(u,v)$ be given by Example 11. The corresponding hamiltonians have the following form:

$$\hat{H}^l = \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}\sum_{k=1,\neq l}^{N}\left(\frac{g^{\alpha\beta}\hat{S}_\alpha^k\hat{S}_\beta^l}{(u_k-u_l)} + \Phi_1^{\alpha\beta}\hat{S}_\alpha^k\hat{S}_\beta^l\right) + \frac{1}{2}\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}\Phi_1^{\alpha\beta}(\hat{S}_\alpha^l\hat{S}_\beta^l + \hat{S}_\beta^l\hat{S}_\alpha^l), \quad l = \overline{1,N}. \tag{39}$$

Here, the tensor $r_0 = \sum_{\alpha=1}^{\dim\mathfrak{g}}\Phi_1^{\alpha\beta}X_\alpha\otimes X_\beta$ is a non-skew nilpotent solution of the constant Yang–Baxter equation. Note that, in this example, no additional constraint was imposed on the numbers $u_k, k = \overline{1,N}$.

Example 14. Let us consider the case $\mathfrak{g} = gl(n), so(n), sp(n)$ and r-matrix $r_\phi(u, v) = r_A(u, v)$ from Example 8. Corresponding hamiltonians have the following explicit form:

$$\hat{H}^l = \sum_{k=1,k\neq l}^{N} \frac{1}{(u_k - u_l)} \sum_{ij=1}^{n} \sqrt{\frac{(1+a_i u_k)(1+a_j u_k)}{(1+a_i u_l)(1+a_j u_l)}} \hat{S}^k_{ij} \hat{S}^l_{ji} + \frac{1}{2} \sum_{ij=1}^{n} \left(\frac{a_i}{(1+a_i u_l)} + \frac{a_j}{(1+a_j u_l)} \right) \hat{S}^l_{ij} \hat{S}^l_{ji}.$$

Here, numbers u_k are subjected to the constraint $(1 + a_i u_k) \neq 0, \forall k \in 1, N, i \in 1, n$. Commutation relations among the "generalized" spin operators $\{\hat{S}^l_{ji}\}$ are standard commutation relations of $gl(n)$, $so(n)$ or $sp(n)$ (see [17]). For example, for $\mathfrak{g} = gl(n)$ we have:

$$[\hat{S}^m_{ij}, \hat{S}^n_{kl}] = \delta^{mn}(\delta_{kj}\hat{S}^n_{il} - \delta_{il}\hat{S}^n_{kj}).$$

Example 15. Let us consider the case $\mathfrak{g} = so(3)$ and r-matrix from Example 12. Corresponding hamiltonians have the following explicit form:

$$\hat{H}^l = \left(\sum_{m=1,m\neq l}^{N} \prod_{i=1}^{3} \sqrt{\frac{(1+a_i u_m)}{(1+a_i u_l)}} \sum_{k=1}^{3} \sqrt{\frac{(1+a_k u_l)}{(1+a_k u_m)}} \frac{\hat{S}^m_k \hat{S}^l_k}{(u_m - u_l)} \right.$$
$$\left. + \frac{1}{2} \prod_{i=1}^{3} (1+a_i u_l)^{-1} \sum_{k=1}^{3} \left(\frac{a_1 a_2 a_3}{a_k} u_l - a_k \right) (1 + a_k u_l) \hat{S}^l_k \hat{S}^l_k \right). \tag{40}$$

Here, numbers u_k are subjected to the constraint $(1 + a_i u_k) \neq 0, \forall k \in 1, N, i \in 1, 3$. Commutation relations among the spin operators $\{\hat{S}^l_k\}$ are standard:

$$[\hat{S}^m_i, \hat{S}^l_j] = \delta^{ml} \sum_{k=1}^{3} \epsilon_{ijk} \hat{S}^l_k. \tag{41}$$

Example 16. Let us consider the previous example in more detail, but in the simplest cases $N = 1$ and $N = 2$. In the case $N = 1$, our construction yields the following hamiltonian:

$$\hat{H}^1 = \frac{1}{2} \prod_{i=1}^{3} (1+a_i u_1)^{-1} \sum_{k=1}^{3} \left(\frac{a_1 a_2 a_3}{a_k} u_1 - a_k \right) (1 + a_k u_1) \hat{S}^1_k \hat{S}^1_k.$$

For $u_1 = 0$, it coincides with the standard quantum hamiltonian of the anisotropic Euler top:

$$\hat{H}^1 = -\frac{1}{2} \sum_{k=1}^{3} a_k \hat{S}^1_k \hat{S}^1_k.$$

In the case $N = 2$, we have two independent commuting quantum hamiltonians:

$$\hat{H}^1 = \prod_{i=1}^{3} \sqrt{\frac{(1+a_i u_2)}{(1+a_i u_2)}} \sum_{k=1}^{3} \sqrt{\frac{(1+a_k u_1)}{(1+a_k u_2)}} \frac{\hat{S}^2_k \hat{S}^1_k}{(u_2 - u_1)}$$
$$+ \frac{1}{2} \prod_{i=1}^{3} (1+a_i u_1)^{-1} \sum_{k=1}^{3} \left(\frac{a_1 a_2 a_3}{a_k} u_1 - a_k \right) (1 + a_k u_1) \hat{S}^1_k \hat{S}^1_k,$$

$$\hat{H}^2 = \prod_{i=1}^{3} \sqrt{\frac{(1+a_i u_1)}{(1+a_i u_2)}} \sum_{k=1}^{3} \sqrt{\frac{(1+a_k u_2)}{(1+a_k u_1)}} \frac{\hat{S}^1_k \hat{S}^2_k}{(u_1 - u_2)}$$
$$+ \frac{1}{2} \prod_{i=1}^{3} (1+a_i u_2)^{-1} \sum_{k=1}^{3} \left(\frac{a_1 a_2 a_3}{a_k} u_2 - a_k \right) (1 + a_k u_2) \hat{S}^2_k \hat{S}^2_k.$$

Let us consider a special choice of u_1, u_2: $u_1 \to 0$, $u_2 \to \infty$. In order for the hamiltonian $\hat{H}^2$ to be non-trivial after this limit, we will consider instead of it the hamiltonian

$$\hat{H}^{2'} = -\lim_{u_2\to\infty}(u_2^2\hat{H}^2 - u_2\hat{c}^2) - \frac{1}{2}\left(\sum_{k=1}^{3} a_k^{-1}\right)\hat{c}^2,$$

where $\hat{c}^2 = \sum_{k=1}^{3} a_k \hat{S}_k^2 \hat{S}_k^2$ is a Casimir operator. We obtain:

$$\hat{H}^1 = \sum_{k=1}^{3}\sqrt{\frac{a_1a_2a_3}{a_k}}\hat{S}_k^2\hat{S}_k^1 - \frac{1}{2}\sum_{k=1}^{3} a_k\hat{S}_k^1\hat{S}_k^1,$$

$$\hat{H}^{2'} = \sum_{k=1}^{3}\sqrt{\frac{a_k}{a_1a_2a_3}}\hat{S}_k^1\hat{S}_k^2 - \frac{1}{2}\sum_{k=1}^{3} a_k^{-1}\hat{S}_k^2\hat{S}_k^2.$$

These hamiltonians are exactly quantized hamiltonians of the Steklov system on $so(4)$ [15].

5. Conclusion and discussion

In the present paper, using general non-skew symmetric classical r-matrices with the spectral parameter, we have constructed new quantum spin chains that generalize famous Gaudin spin chains. With the help of the special quasigraded Lie algebras, we have explicitly constructed new non-skew symmetric classical r-matrices with the spectral parameters. Using them, we have obtained a number of new examples of integrable quantum Gaudin-type spin chains.

An interesting and important problem is the diagonalization of these hamiltonians, i.e. explicit construction of their eigen-vectors and eigen-values. Work on this problem is now in progress; the first results are already obtained and we will return to them in future publications.

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Appendix. Proof of the main theorem

We will prove the theorem directly showing that $[\hat{H}^k, \hat{H}^l] = 0\ \forall k, l, k \neq l$. For this purpose, we represent our hamiltonians as follows: $\hat{H}^k = \hat{H}_0^k + \hat{H}_1^k$, $\hat{H}^l = \hat{H}_0^l + \hat{H}_1^l$, where

$$\hat{H}_0^l = \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}}\sum_{k\neq l}^{N} r^{\alpha\beta}(u_k, u_l)\hat{S}_\alpha^k\hat{S}_\beta^l, \quad \hat{H}_1^l = 1/2\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} r_0^{\alpha\beta}(u_l, u_l)(\hat{S}_\alpha^l\hat{S}_\beta^l + \hat{S}_\beta^l\hat{S}_\alpha^l).$$

It is easy to see that $[\hat{H}_1^k, \hat{H}_1^l] = 0$, and hence:

$$[\hat{H}^k, \hat{H}^l] = [\hat{H}_0^k, \hat{H}_0^l] + [\hat{H}_0^k, \hat{H}_1^l] + [\hat{H}_1^k, \hat{H}_0^l]. \tag{42}$$

Let us calculate each of these summands separately:

$$\begin{aligned}[\hat{H}_0^k, \hat{H}_0^l] &= \sum_{m\neq k}\sum_{n\neq l}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_m, u_k)r^{\gamma,\delta}(u_n, u_l)[\hat{S}_\alpha^m\hat{S}_\beta^k, \hat{S}_\gamma^n\hat{S}_\delta^l] \\ &= \sum_{m,n\neq k,l}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_m, u_k)r^{\gamma,\delta}(u_n, u_l)[\hat{S}_\alpha^m\hat{S}_\beta^k, \hat{S}_\gamma^n\hat{S}_\delta^l] \\ &\quad + \sum_{n\neq l,k}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_l, u_k)r^{\gamma,\delta}(u_n, u_l)[\hat{S}_\alpha^l\hat{S}_\beta^k, \hat{S}_\gamma^n\hat{S}_\delta^l]\end{aligned}$$

$$+\sum_{m\neq k,l}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_m,u_k)r^{\gamma,\delta}(u_k,u_l)[\hat{S}^m_\alpha \hat{S}^k_\beta, \hat{S}^k_\gamma \hat{S}^l_\delta]$$
$$+\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_l,u_k)r^{\gamma,\delta}(u_k,u_l)[\hat{S}^l_\alpha \hat{S}^k_\beta, \hat{S}^k_\gamma \hat{S}^l_\delta]$$
$$=\sum_{m\neq k,l} r^{\alpha,\beta}(u_m,u_k)r^{\gamma,\delta}(u_m,u_l)C^\kappa_{\alpha\gamma}\hat{S}^m_\kappa \hat{S}^k_\beta \hat{S}^l_\delta + \sum_{m\neq l,k}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_l,u_k)r^{\gamma,\delta}(u_m,u_l)C^\kappa_{\alpha\delta}\hat{S}^l_\kappa \hat{S}^k_\beta \hat{S}^m_\gamma$$
$$+\sum_{m\neq k,l}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_m,u_k)r^{\gamma,\delta}(u_k,u_l)C^\kappa_{\beta\gamma}\hat{S}^m_\alpha \hat{S}^k_\kappa \hat{S}^l_\delta$$
$$+\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_l,u_k)r^{\gamma,\delta}(u_k,u_l)[\hat{S}^l_\alpha \hat{S}^k_\beta, \hat{S}^k_\gamma \hat{S}^l_\delta].$$

Note that, in the first three summands of the final expression in the right-hand side of this equality, indices m, k, l are all different and the order of the following operators $\hat{S}^m_\kappa$, $\hat{S}^k_\beta$, $\hat{S}^l_\delta$ is not important. Let us consider these summands in more details. Taking into account that $r_{12}(u_m,u_k) = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha\beta}(u_m,u_k)X_\alpha \otimes X_\beta \otimes 1$, $r_{13}(u_m,u_l) = \sum_{\gamma,\delta=1}^{\dim \mathfrak{g}} r^{\gamma\delta}(u_m,u_l)X_\gamma \otimes 1 \otimes X_\delta$, $r_{23}(u_k,u_l) = \sum_{\gamma,\delta=1}^{\dim \mathfrak{g}} r^{\gamma\delta}(u_k,u_l)1 \otimes X_\gamma \otimes X_\delta$, and $r_{32}(u_l,u_k) = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} r^{\alpha\beta}(u_k,u_l)1 \otimes X_\beta \otimes X_\alpha$, we obtain that they can be re-written in the following way:

$$\langle [r_{12}(u_m,u_k), r_{13}(u_m,u_l)] + [r_{12}(u_m,u_k), r_{23}(u_k,u_l)] + [r_{32}(u_l,u_k), r_{13}(u_m,u_l)], \hat{S}^m_1 \hat{S}^k_2 \hat{S}^l_3\rangle$$

where we have used the following notations:

$$\hat{S}^m_1 = \sum_{\alpha=1}^{\dim \mathfrak{g}} \hat{S}^m_\alpha X^\alpha \otimes 1 \otimes 1, \quad \hat{S}^k_2 = \sum_{\beta=1}^{\dim \mathfrak{g}} 1 \otimes \hat{S}^m_\beta X^\beta \otimes 1, \quad \hat{S}^l_3 = \sum_{\delta=1}^{\dim \mathfrak{g}} 1 \otimes 1 \otimes \hat{S}^m_\delta X^\delta,$$

and $\langle\ ,\ \rangle$ is a scalar product on $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ extended in a natural way from the scalar product on $\mathfrak{g}$ and $\langle X_\alpha, X^\beta\rangle = \delta^\beta_\alpha$. By virtue of the generalized Yang–Baxter equation, these summands are canceled. Hence, we obtain:

$$[\hat{H}^k_0, \hat{H}^l_0] = \sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_l,u_k)r^{\gamma,\delta}(u_k,u_l)[\hat{S}^l_\alpha \hat{S}^k_\beta, \hat{S}^k_\gamma \hat{S}^l_\delta].$$

Let us consider this expression in detail:

$$[\hat{S}^l_\alpha \hat{S}^k_\beta, \hat{S}^k_\gamma \hat{S}^l_\delta] = \hat{S}^l_\alpha \hat{S}^l_\delta[\hat{S}^k_\beta, \hat{S}^k_\gamma] + [\hat{S}^l_\alpha, \hat{S}^l_\delta]\hat{S}^k_\gamma \hat{S}^k_\beta, \tag{43}$$

where we have again used that $\hat{S}^l_\delta$ and $\hat{S}^k_\beta$ mutually commute. Let us show that this expression is symmetric with respect to the permutation of all operators. It is enough to show that it is symmetric with respect to the permutation of the operators with the same upper indices. We have:

$$[\hat{S}^l_\alpha \hat{S}^k_\beta, \hat{S}^k_\gamma \hat{S}^l_\delta] = \frac{1}{2}(\hat{S}^l_\alpha \hat{S}^l_\delta + \hat{S}^l_\delta \hat{S}^l_\alpha)[\hat{S}^k_\beta, \hat{S}^k_\gamma] + \frac{1}{2}[\hat{S}^l_\alpha, \hat{S}^l_\delta](\hat{S}^k_\gamma \hat{S}^k_\beta + \hat{S}^k_\beta \hat{S}^k_\gamma)$$
$$+\frac{1}{2}([\hat{S}^l_\alpha, \hat{S}^l_\delta][\hat{S}^k_\beta, \hat{S}^k_\gamma] + [\hat{S}^l_\alpha, \hat{S}^l_\delta][\hat{S}^k_\gamma, \hat{S}^k_\beta])$$
$$=\frac{1}{2}C^\kappa_{\beta\gamma}(\hat{S}^l_\alpha \hat{S}^l_\delta + \hat{S}^l_\delta \hat{S}^l_\alpha)\hat{S}^k_\kappa + \frac{1}{2}C^\kappa_{\alpha\delta}\hat{S}^l_\kappa(\hat{S}^k_\gamma \hat{S}^k_\beta + \hat{S}^k_\beta \hat{S}^k_\gamma).$$

We obtain that formula (43) can be re-written in the symmetrized tensor form:

$$[\hat{H}^k_0, \hat{H}^l_0] = \frac{1}{2}\langle [r_{12}(u_l,u_k), r_{23}(u_k,u_l)], ((\hat{S}^l_1 \hat{S}^l_3))\hat{S}^k_2\rangle + \frac{1}{2}\langle [r_{32}(u_l,u_k), r_{13}(u_k,u_l)], ((\hat{S}^k_1 \hat{S}^k_2))\hat{S}^l_3\rangle, \tag{44}$$

where $((\hat{S}^l_1 \hat{S}^l_3)) \equiv \sum_{\alpha\beta}(\hat{S}^l_\alpha \hat{S}^l_\beta + \hat{S}^l_\beta \hat{S}^l_\alpha)X_\alpha \otimes 1 \otimes X_\beta$ and $((\hat{S}^k_1 \hat{S}^k_2)) \equiv \sum_{\alpha\beta}(\hat{S}^k_\alpha \hat{S}^k_\beta + \hat{S}^k_\beta \hat{S}^k_\alpha)X_\alpha \otimes X_\beta \otimes 1$.

Let us now calculate the expression $[H_1^k, H_0^l]$ explicitly:

$$
\begin{aligned}
[\hat{H}_1^k, \hat{H}_0^l] &= \frac{1}{2}\sum_{m\neq k}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_m,u_k) r_0^{\gamma,\delta}(u_l,u_l)[\hat{S}_\alpha^m \hat{S}_\beta^k, (\hat{S}_\gamma^l \hat{S}_\delta^l + \hat{S}_\delta^l \hat{S}_\gamma^l)] \\
&= \frac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_l,u_k) r_0^{\gamma,\delta}(u_l,u_l)[\hat{S}_\alpha^l \hat{S}_\beta^k, (\hat{S}_\gamma^l \hat{S}_\delta^l + \hat{S}_\delta^l \hat{S}_\gamma^l)] \\
&= \frac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} r^{\alpha,\beta}(u_l,u_k) r_0^{\gamma,\delta}(u_l,u_l)\hat{S}_\beta^k (C_{\alpha\delta}^\kappa(\hat{S}_\kappa^l \hat{S}_\gamma^l + \hat{S}_\kappa^l \hat{S}_\gamma^l) + C_{\alpha\gamma}^\kappa(\hat{S}_\kappa^l \hat{S}_\delta^l + \hat{S}_\kappa^l \hat{S}_\delta^l)) \\
&= \frac{1}{2}\langle[r_{32}(u_l,u_k) + r_{12}(u_l,u_k), (r_0)_{13}(u_l,u_l)], ((\hat{S}_1^l \hat{S}_3^l))\hat{S}_2^k\rangle.
\end{aligned}
\tag{45}
$$

In an analogous way, we obtain:

$$
[\hat{H}_0^k, \hat{H}_1^l] = \frac{1}{2}\langle[(r_0)_{12}(u_k,u_k), r_{13}(u_k,u_l) + r_{23}(u_k,u_l)], ((\hat{S}_1^k \hat{S}_2^k))\hat{S}_3^l\rangle. \tag{46}
$$

Let us now show that the expression $[\hat{H}_0^k, \hat{H}_0^l]$ cancels the expressions $[\hat{H}_1^k, \hat{H}_0^l] + [\hat{H}_0^k, \hat{H}_1^l]$. For this purpose, we will more explicitly calculate the expressions $[r_{12}(u_l,u_k), r_{23}(u_k,u_l)]$ and $[r_{32}(u_l,u_k), r_{13}(u_k,u_l)]$ using the generalized classical Yang–Baxter equation:

$$
\begin{aligned}
[r_{12}(u_l,u_k), r_{23}(u_k,u_l)] &= \lim_{u_m\to u_l}([r_{13}(u_l,u_m), r_{12}(u_l,u_k) + r_{32}(u_m,u_k)]) \\
&= \lim_{u_m\to u_l}\left(\left[\frac{\widehat{C}_{13}}{(u_l-u_m)} + (r_0)_{13}(u_m,u_l), r_{12}(u_l,u_k) + r_{32}(u_l,u_k)\right.\right. \\
&\quad \left.\left. + \sum_{s=1}^{\infty}\frac{1}{s!}\partial_{u_l}^s(r_{32}(u_l,u_k))(u_m-u_l)^s\right]\right).
\end{aligned}
$$

It is easy to show that $[\widehat{C}_{13}, r_{12}(u_l,u_k) + r_{32}(u_l,u_k)] = 0$, and we obtain:

$$
[r_{12}(u_l,u_k), r_{23}(u_k,u_l)] = [(r_0)_{13}(u_l,u_l), r_{12}(u_l,u_k) + r_{32}(u_l,u_k)] - [\widehat{C}_{13}, \partial_{u_l} r_{32}(u_l,u_k)]. \tag{47}
$$

In an analogous way, it is easy to show, that:

$$
[r_{32}(u_l,u_k), r_{13}(u_k,u_l)] = [r_{13}(u_k,u_l) + r_{23}(u_k,u_l), (r_0)_{12}(u_k,u_k)] + [\widehat{C}_{12}, \partial_{u_k} r_{23}(u_k,u_l)]. \tag{48}
$$

Substituting equalities (47) and (48) into expression (44) and then substituting (44) together with the expressions (45) and (46) into Eq. (42), we obtain:

$$
[\hat{H}^k, \hat{H}^l] = \langle[\widehat{C}_{12}, \partial_{u_k} r_{23}(u_k,u_l)]((\hat{S}_1^k \hat{S}_2^k))\hat{S}_3^l\rangle - \langle[\widehat{C}_{13}, \partial_{u_l} r_{32}(u_l,u_k)]((\hat{S}_1^l \hat{S}_3^l))\hat{S}_2^k\rangle.
$$

On the other hand:

$$
\begin{aligned}
[\widehat{C}_{13}, \partial_{u_l} r_{32}(u_l,u_k)] &= \sum_{\alpha,\beta,\gamma,\delta} g^{\alpha,\beta}\partial_{u_l} r^{\delta,\gamma}(u_l,u_k)[X_\alpha \otimes 1 \otimes X_\beta, 1 \otimes X_\gamma \otimes X_\delta] \\
&= \sum_{\alpha,\beta,\gamma,\delta,\kappa} g^{\alpha,\beta} C_{\beta\delta}^\kappa \partial_{u_l} r^{\delta,\gamma}(u_l,u_k) X_\alpha \otimes X_\gamma \otimes X_\kappa \\
&= \sum_{\alpha,\gamma,\delta,\kappa} C_\delta^{\kappa\alpha}\partial_{u_l} r^{\delta,\gamma}(u_l,u_k) X_\alpha \otimes X_\gamma \otimes X_\kappa.
\end{aligned}
$$

It is easy to see that this expression is skew symmetric in the first and third tensor indices. In an analogous way, it is shown that $[\widehat{C}_{12}, \partial_{u_k} r_{23}(u_k,u_l)]$ is skew-symmetric in the first and second tensor indices. Hence, after convolution with $((\hat{S}_1^l \hat{S}_3^l))$ and $((\hat{S}_1^k \hat{S}_2^k))$ correspondingly, this expression turns to zero. Hence $[\hat{H}^k, \hat{H}^l] = 0$.

The theorem is proved.

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